Lecture 12 on Oct. 24 2013

In the last lecture, we have seen that

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right)$$

maps the region $A = \{\theta_0 < \theta < \pi - \theta_0\}$ to the region B, which is the region between two branches of the hyperbola

$$\frac{x^2}{\cos^2\theta_0} - \frac{y^2}{\sin^2\theta_0} = 1.$$

Moreover e^z maps the strip $L = \{\theta_0 < y < \pi - \theta_0\}$ to the region A. Therefore composing these two mappings together, we know that $\cosh z$ maps L to B. By rotation mapping, we know that $L_1 = \{\theta_0 < x < \pi - \theta_0\}$ can be mapped to L via the mapping iz. Therefore composing iz with $\cosh z$, we know that $\cos z$ maps L_1 to B. By translation, we know that $L_2 = \{\theta_0 - \pi/2 < x < \theta_0 + \pi/2\}$ can be mapped to L_1 via the mapping $z + \pi/2$. Therefore composing $\cos z$ with $z + \pi/2$, we know that $\sin z$ maps L_2 to B. Up to now, we show that sinh, cosh, sin and cos functions are essentially a same one. They only differ with each other by rotation and translation.

In the following, we consider sin function and study the image of $\sin z$ when acts on the semi-infinite strip $L^* = \{-\pi/2 < x < \pi/2, y > 0\}$. Fixing $x = \theta_0 - \pi/2$, where θ_0 lies in $(0, \pi/2)$, then we know that the point $\theta_0 - \pi/2 + iy$ is mapped to

$$-\frac{1}{2}\cos\theta_0\left(e^{-y} + e^{y}\right) - \frac{i}{2}\sin\theta_0\left(e^{-y} - e^{y}\right).$$

While y runs from 0 to ∞, the x-coordinate for the imaging point runs from $-\cos\theta_0$ to $-\infty$. The y-coordinate for the imaging point runs from 0 to ∞. Same arguments can be applied to the semi-straight line $\{x = \theta_0 + \pi/2, y > 0\}$. By the above arguments, one can easily show that L^* is mapped to the upper half plane via the sin function.

Up to now, we have considered mappings from strips, regions between two rays and semi-infinite strips. For strips, the boundary are two parallel lines. When we look from the Riemann sphere point of view, two parallel lines are circles with only one intersection at north pole. As for regions between two rays, the boundary are two semi straight lines with an intersection at the origin. From the Riemann sphere point of view, the boundaries are formed by two circles with two intersections. One intersection is the origin and another one is the ∞ (north pole). Basically we can also map regions between two circles on $\mathbb{C}$. In fact, given two circles on $\mathbb{C}$, they have at most two intersections. If they have two, say z_0 and z_1, are two intersections, then we can define linear transformation $(z - z_0)/(z - z_1)$. Clearly this mapping maps z_0 to 0 and maps z_1 to ∞. two circles are mapped to two straight lines. Therefore, the region enclosed by these two circles are mapped to regions between two rays. If there is just one intersection, say z_0, then one can use $1/(z - z_0)$ so that the region between the two circles are mapped to an infinite strip. Thanks to Henry. He pointed this case to me via our email correspondence.